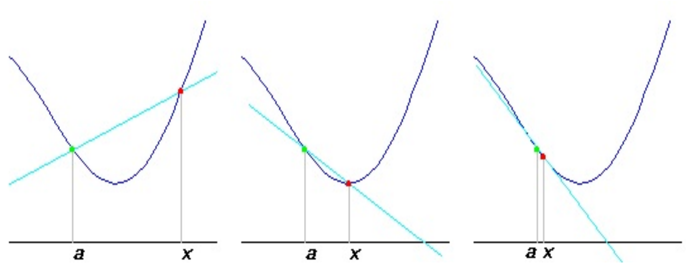
Mathematics Tutorial Series

Differential Calculus Video 5

Differentiation: Definition and Notation

The derivative of a function *y = f(x)* at a point *x = a* is the **rate of change** of the function at the point *x = a*. Geometrically, this means that the value of the derivative of *f(x)* at *x = a* equals the **slope of the tangent line** of the graph of *y = f(x)* at the point *x = a*.

There is a formal definition of the derivative.



As the point x moves toward a, the secant comes closer to being a tangent.

The secant slope is $\frac{Rise}{Run} = \frac{f(x)-f(a)}{x-a}$

Let x go to a. Then we have: $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

This is the formal definition of the value of the derivative at the point a.

Notice that this limit always looks like $\frac{0}{0}$ which is **meaningless.**

Somehow we have to simplify the fraction $\frac{f(x)-f(a)}{x-a}$ **before** taking the limit.

Example 1:

Let $y = f(x) = x^2$.

The value of the derivative of f at a is:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{x^2 - a^2}{x - a}$$

$$= \lim_{x \to a} \frac{(x + a)(x - a)}{x - a}$$

$$= \lim_{x \to a} x + a$$

$$= 2\,a$$

Notation

If $y = f(x)$ then the derivative with respect to x is written in several different ways. The choice depends on the author and the circumstance.

$$\frac{dy}{dx} = \frac{df}{dx} = \frac{df(x)}{dx} = y' = f'$$

Example: $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

If the variable is t then this changes to:

$$\frac{dy}{dt} = \frac{df}{dt} = \frac{df(t)}{dt} = y' = f'$$

Hence, from our Example 1: $\frac{dx^2}{dx} = 2\,x$

Example 2: $y = f(x) = cx$ for a constant number c

Then:

$$\frac{dy}{dx}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{cx - ca}{x - a}$$

$$= \lim_{x \to a} \frac{c(x - a)}{x - a}$$

$$= \lim_{x \to a} c$$

$$= c$$

Alternate Definition

We have defined the derivative of function $f(x)$ at point a as the limit:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

If we use $h = x - a$ to measure the distance from x to a, then the limit can be written as:

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

This is often a more convenient way to calculate the derivative.

Use of the Formal Definition

The formal definition is used rarely.

It is used to prove that the **basic rules** are true:

- Sum Rule
- Product Rule
- Chain Rule

It is used to calculate a few **basic derivatives**:

$$c, x, \sin x \text{ and } e^x.$$

Summary:

- There is a formal definition of a derivative that uses a limit:
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
- If we use $h = x - a$ to measure the distance from x to a, then the limit can be written as:
    - $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$
- The formal definition is used very rarely:
    - To prove that the sum, product and chain rules work;
    - To calculate the basic derivatives $c, x, \sin x$ and e^x.

Exercise:

Use the formal definition to show that $\frac{dx}{dx} = 1$